Radiation Transfer in Isotropically Scattering, Rectangular Enclosures

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Radiative heat transfer in absorbing, emitting, isotropically scattering, gray, two-dimensional rectangular regions having an emitting boundary and spacially distributed energy sources is solved by both the collocation and the Galerkin methods. Using the expressions given in this work, the quantities, such as the incident radiation, the radiation heat flux, and the divergence of the radiation heat flux anywhere in the medium, can be determined to a high degree of accuracy for all values of the single scattering albedo, from small to moderately large values of horizontal and vertical optical dimensions of the enclosure. Results are presented for representative cases, to illustrate the application of the method. A parameter study is made to show the effects of the single scattering albedo on the incident radiation and the heat flux.

Introduction

THERE are numerous situations in which radiation heat transfer in absorbing, emitting, isotropically scattering, gray, two-dimensional rectangular media is important. They include, among others, modeling of heat transfer in furnaces, fire protection, manufacturing of glass, studies of insulation properties of various types of fibrous and foam materials, cryogenics, and heat transfer through partially transparent materials.

In order to study radiation heat transfer in any one of these situations, it is necessary to solve the equation of radiative heat transfer. Numerous approximate and exact methods of analysis have been reported for solving the equation of radiative transfer in a one-dimensional medium; but for the two-dimensional situation, the problem of an absorbing and emitting medium has been studied with approximate methods of analysis by several investigators including Modest,[1] Razzaque et al.,[2] and Fiveland.[3] In the case of a scattering medium, the source function expansion technique has been used by Sutton and Özisik[4]; a numerical integration scheme has been employed by Crosbie and Schrenker[5] to study the problem of diffuse incident radiation and no energy sources in the medium; and the same numerical technique has been applied by Crosbie and Schrenker[6] to solve the problem of collimated incident radiation and no energy sources in the medium. Further information, with regard to solution methods and areas of applications of radiative heat transfer, is given in a recent survey by Viskanta.[7]

We present a highly accurate method for solving two-dimensional radiation transfer in an absorbing, emitting, isotropically scattering, two-dimensional rectangular enclosure. In the following sections, the method of analysis is described first. Results are then presented that can be used to check the accuracy of various approximate methods of analysis. Finally, a parameter survey is made to examine the effects of single scattering albedo on the radiation heat transfer for various optical dimensions of the medium.

Problem Formulation

We consider an obsorbing, emitting, isotropically scattering, gray, homogeneous, rectangular enclosure confined to the domain $-b \leq y \leq b$, $-c \leq z \leq c$, as illustrated in Fig. 1. The medium has spatially distributed energy sources of intensity $I_b[T(y,z)]$, which vary with position. The boundary surface at $z=-c$ is opaque and emits energy of unit intensity isotropically. The remaining three boundary surfaces are considered transparent and contain no external incident energy sources. The modeling of the physical situation is equivalent to considering the bottom surface $z=-c$ as black with unit emissive power, while the other three surfaces are black with zero emissive power (cold). The mathematical formulation to this problem is obtainable by performing an energy balance on a cylindrical volume element aligned in the direction of the propagating radiation intensity (e.g., see Ref. 8). By changing the directional derivative into partial derivatives, one obtains the following linear, inhomogeneous, partial integrodifferential equation

$$\sin\theta\sin\phi\frac{\partial I(y,z,\theta,\phi)}{\partial y}+\cos\theta\frac{\partial I(y,z,\theta,\phi)}{\partial z}$$

$$+I(y,z,\theta,\phi)=(1-\omega)I_b[T(y,z)]+\frac{\omega}{4\pi}G(y,z)$$

$$\text{in } |z|<c,\quad |y|<b,\quad 0\leq\theta\leq\pi,\quad -\pi\leq\phi\leq\pi \tag{1}$$

and the boundary conditions are of the forms

$$I(y,-c,\theta,\phi)=1,\quad -b<y<b,\quad 0\leq\theta\leq\frac{\pi}{2},\quad -\pi\leq\phi\leq\pi \tag{2a}$$

$$I(y,c,\theta,\phi)=0,\quad |y|\leq b,\quad \frac{\pi}{2}\leq\theta\leq\pi,\quad -\pi\leq\phi\leq\pi \tag{2b}$$

$$I(-b,z,\theta,\phi)=0,\quad -c<z\leq c,\quad 0\leq\theta\leq\pi,\quad 0\leq\phi\leq\pi \tag{2c}$$

$$I(b,z,\theta,\phi)=0,\quad -c<z<c,\quad 0<\theta\leq\pi,\quad -\pi\leq\phi\leq 0 \tag{2d}$$

Here $I(y,z,\theta,\phi)$ is the radiation intensity, y and z are the optical variables, θ is the polar angle, and ϕ is the azimuthal angle as shown in Fig. 2. ω is the single scattering albedo, $I_b[T(y,z)]$ is the spatially distributed energy source due to the temperature $T(y,z)$ of the medium, and $G(y,z)$ is the incident radiation defined as

$$G(y,z)=\int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi}I(y,z,\theta,\phi)\sin\theta\mathrm{d}\phi\mathrm{d}\theta \tag{3}$$

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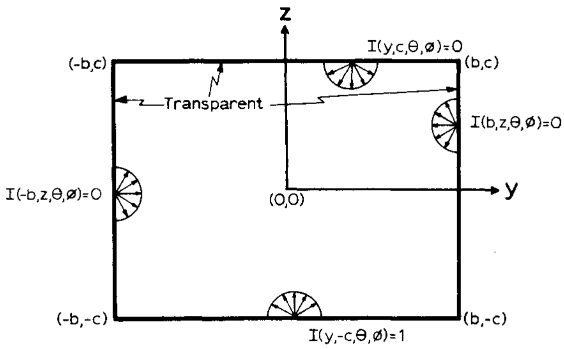

Fig. 1 Geometry, coordinates, and boundary conditions.

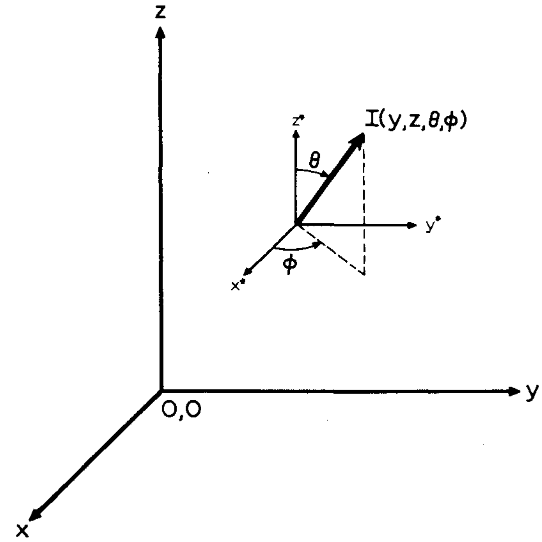

Fig. 2 Angular variables.

Equation (1) is of the hyperbolic type, and as such possesses characteristic lines along which discontinuities are propagated across the solution domain of the intensity. Due to the form of the boundary conditions in Eqs. (2), characteristic lines intersect the corners at $z=-c$, $y=\pm b$ $(0<\theta<\pi/2)$. However, the integration of the intensity over all solid angles in Eq. (3) has the effect of smoothing out such discontinuities. We now describe the development of the integral form of Eq. (1). As shown by Smith,[9] Eq. (1) is replaced with four first-order differential equations by introducing two new independent variables. By treating these four equations as first-order ordinary differential equations, a direct integration yields formal solutions to the intensities (i.e., the integration is performed along a ray of intensity that propagates between the boundary surface and the point y,z). Since each of these solutions is defined for specified limits on θ and ϕ, the integral defined by Eq. (3) is split up into the same limits. Hence, by substituting the four formal solutions into Eq. (3), and by utilizing an appropriate transformation, one arrives at the integral equation for the function $G(y,z)$

$$G(y,z)=2f_{0,0}\left[\frac{1}{2}(b+y),\frac{1}{2}(c+z)\right]+2f_{0,0}\left[\frac{1}{2}(b-y),\frac{1}{2}(c+z)\right]+2\int_{v=-c}^{c}\int_{u=-b}^{b}K_1(u,y;v,z)\left\{(1-\omega)I_b[T(u,v)]+\frac{\omega}{4\pi}G(u,v)\right\}\mathrm{d}u\mathrm{d}v \tag{4}$$

where the functions $f_{0,0}(u,v)$ and $K_1(u,y;v,z)$ are special cases of the functions $f_{i,j}(u,v)$ and $K_n(u,y;v,z)$, defined as

$$f_{i,j}(u,v)=\int_{\theta=0}^{\tan^{-1}(u/v)}\cos^i\theta\tan^j\theta Ki_{i+2}(2v\sec\theta)\mathrm{d}\theta \tag{5a}$$

$$K_n(u,y;v,z)=Ki_n\{[(u-y)^2+(v-z)^2]^{1/2}\}[(u-y)^2+(v-z)^2]^{-n/2} \tag{5b}$$

$Ki_n(x)$ is the Bickley function,[10] which is an integral of the form

$$Ki_n(x)=\int_0^{\pi/2}e^{-x\sec\theta}\cos^{n-1}\theta\mathrm{d}\theta \tag{6}$$

The advantage of the integral form given by Eq. (4) is now apparent—the integral equation involves two independent variables whereas Eq. (1) has four independent variables. The reduction of independent variables from 4 to 2 represents a great savings in the solution of the radiation problem considered here. However, the kernel of the integral equation has a singularity at $u=y$, $v=z$ that has to be carefully dealt with. Once a solution to the function $G(y,z)$ is obtained, all other quantities of interest, such as the partial heat flux vectors, the divergence of the heat flux vector, and the radiation intensity anywhere in the medium are determined readily from their respective definitions, in terms of $G(y,z)$.

Method of Solution

To solve the integral equation (4), it is desirable to base the method of solution on the appropriate eigenfunctions of the homogeneous form of Eq. (1). Although such eigenfunctions are available for the problem of radiation heat transfer in the plane parallel case,[8] they are not available, to our knowledge, for the present two-dimensional situation. Instead we assume a power series representation of the function $G(y,z)$

$$G(y,z)=\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}y^iz^j \tag{7a}$$

where C_{ij} are $(I+1)(J+1)$ unknown expansion coefficients We assume that the distributed energy source $I_b[T(y,z)]$ can also be represented by a power series in the form

$$I_b[T(y,z)]=\sum_{k=0}^{K}\sum_{\ell=0}^{L}A_{k\ell}y^kz^\ell \tag{7b}$$

where $A_{k\ell}$ are $(K+1)(L+1)$ known expansion coefficients. Using a power series expansion for both $G(y,z)$ and $I_b[T(y,z)]$ has the advantage that all ensuing integrals are identical; thus, a significant reduction in computer time is achieved. Equations (7) are substituted into Eq. (4) to obtain

$$\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}\left[y^iz^j-\frac{\omega}{2\pi}T_{ij}(y,z)\right]=2f_{0,0}\left[\frac{1}{2}(b+y),\frac{1}{2}(c+z)\right]+2f_{0,0}\left[\frac{1}{2}(b-y),\frac{1}{2}(c+z)\right]+2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L}A_{k\ell}T_{k\ell}(y,z) \tag{8}$$

where the integral term $T_{ij}(y,z)$ is defined in Appendix A, and an explicit expression is developed.

We now are concerned with the determination of the unknown coefficients C_{ij}. We consider two different approaches for obtaining a solution to the expansion coefficients C_{ij}: 1) the collocation method and 2) the Galerkin method.

In the collocation method, one chooses $(I+1)(J+1)$ collocation points y_m $(m=0,...,I)$ and z_n $(n=0,...,J)$ at which the error involved in the expansion of $GF(y,z)$ is forced to vanish.[11] This results in the following $(I+1)$ $(J+1)$ linear algebraic equations for the unknown expansion coefficients C_{ij}

$$\sum_{i=0}^{I}\sum_{j=0}^{J} C_{ij}\left[y_m^i z_n^j - \frac{\omega}{2\pi}T_{ij}(y_m,z_n)\right] = 2f_{0,0}\left[\frac{1}{2}(b+y_m),\frac{1}{2}(c+z_n)\right] + 2f_{0,0}\left[\frac{1}{2}(b-y_m),\frac{1}{2}(c+z_n)\right] + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}T_{k\ell}(y_m,z_n)$$

$$m=0,...,I, \qquad n=0,...,J \tag{9}$$

However, the strategy for choosing the collocation points, such that a rapid rate of convergence of the solution is guaranteed, is not well understood.[12] It appears that a rapid rate of convergence of the solution is obtained by choosing the collocation points as the zeroes of the Chebyshev polynomials as applied to a particular integral equation by Sloan and Burn.[13] The method of equally spaced collocation points has been applied to another type of integral equation with satisfactory results.[12] One disadvantage of using the zeroes of the Chebyshev polynomials as the collocation points is that the integrals need to be reevaluated for each change in the order of the expansion.

In the Galerkin method, Eq. (8) is operated on by the double integral

$$\int_{-c}^{c}\int_{-b}^{c} y^\mu z^\nu \mathrm{d}y\mathrm{d}z, \quad \mu=0,...,I, \quad \nu=0,...,J$$

with the requirement that the error involved in the expansion of $G(y,z)$ is orthogonal to the functions y^μ and z^ν (see Refs. 11 and 14). This yields $(I+1)(J+1)$ linear algebraic equations for the unknown expansion coefficients C_{ij}, which are written in matrix form

$$[b_{ij}^{\mu\nu}]\{C_{ij}\} = \{\mathrm{d}^{\mu\nu}\}, \qquad \mu=0,...,I, \quad \nu=0,...,J \tag{10}$$

where the elements $b_{ij}^{\mu\nu}$ and $\mathrm{d}^{\mu\nu}$ are defined by

$$b_{ij}^{\mu\nu} = \frac{b^{i+\mu+1}c^{j+\nu+1}}{(i+\mu+1)(j+\nu+1)}[1+(-1)^{i+\mu}] \times [1+(-1)^{j+\nu}] - \frac{\omega}{2\pi}T_{ij}^{\mu\nu} \tag{11a}$$

and

$$\mathrm{d}^{\mu\nu} = H^{\mu\nu} + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L}A_{k\ell}T_{k\ell}^{\mu\nu} \tag{11b}$$

where the functions $T_{ij}^{\mu\nu}$ and $H^{\mu\nu}$ are integrals defined in Appendix B and for which explicit expressions are developed. We note matrix $[b_{ij}^{\mu\nu}]$ is symmetric and elements $b_{ij}^{\mu\nu}$ are zero when either $i+\mu$ or $j+\nu$ is odd. The Galerkin method has been successfully applied to radiation heat transfer problems in one-dimensional media.[15] It is clear, however, that the success of the method is measured in the efficiency and accuracy of the evaluation of the integrals.

Once the expansion coefficients are determined, either by the collocation method or the Galerkin method, the incident radiation is computed from

$$G(y,z) = 2f_{0,0}\left[\frac{1}{2}(b+y),\frac{1}{2}(c+z)\right] + 2f_{0,0}\left[\frac{1}{2}(b-y),\frac{1}{2}(c+z)\right] + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}T_{k\ell}(y,z) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}T_{ij}(y,z) \tag{12}$$

As the effects of the boundary surface and internal sources are included exactly in Eq. (12), this is a much more accurate expression than Eq. (7a) for the same order of expansion.

The forward and backward partial heat fluxes in the z direction, $q_z^+(y,z)$ and $q_z^-(y,z)$, and the forward and backward partial heat fluxes in the y direction, $q_z^+(y,z)$ and $q_y^-(y,z)$, are determined, respectively, from

$$q_z^+(y,z) = 2f_{1,0}\left[\frac{1}{2}(b+y),\frac{1}{2}(c+z)\right] + 2f_{1,0}\left[\frac{1}{2}(b-y),\frac{1}{2}(c+z)\right] + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}Z_{k\ell}^+(y,z) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}Z_{ij}^+(y,z) \tag{13a}$$

$$q_z^-(y,z) = 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L}A_{k\ell}Z_{k\ell}^-(y,z) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}Z_{ij}^-(y,z) \tag{13b}$$

$$q_y^+(y,z) = 2f_{1,1}\left[\frac{1}{2}(b+y),\frac{1}{2}(c+z)\right] + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L}A_{k\ell}Y_{k\ell}^+(y,z) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}Y_{ij}^+(y,z) \tag{13c}$$

and

$$q_y^-(y,z) = 2f_{1,1}\left[\frac{1}{2}(b-y),\frac{1}{2}(c+z)\right] + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L}A_{k\ell}Y_{k\ell}^-(y,z) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J}C_{ij}Y_{ij}^-(y,z) \tag{13d}$$

where the functions $Z_{ij}^\pm(y,z)$ and $Y_{ij}^\pm(y,z)$ are defined in Appendix C.

The total heat flow rates $Q_z^\pm(\pm c)$ and $Q_y^\pm(\pm b)$ at boundary surfaces $z=\pm c$ and $y=\pm b$ are defined

$$Q_z^\pm(\pm c) = \int_{-b}^{b} q_z^\pm(y,\pm c)\mathrm{d}y \tag{14a}$$

and

$$Q_y^\pm(\pm b) = \int_{-c}^{c} q_y^\pm(\pm b,z)\mathrm{d}z \tag{14b}$$

Introducing Eqs. (13) into Eqs. (14), the explicit expressions for the heat flow rates at the boundary surfaces become

$$Q_z^+(c) = 8bf_{1,0}(b,c) - 8cf_{1,1}(b,c) + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}W_{k\ell}^+(b,c) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J} C_{ij}W_{ij}^+(b,c) \tag{15a}$$

$$Q_z^-(-c) = 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}W_{k\ell}^-(b,c) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J} C_{ij}W_{ij}^-(b,c) \tag{15b}$$

$$Q_y^+(b) = g_{0,3} - 2f_{2,1}(b,c) - 2f_{2,1}(c,b) + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}W_{\ell k}^+(c,b) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J} C_{ij}W_{ji}^+(c,b) \tag{15c}$$

and

$$Q_y^-(-b) = g_{0,3} - 2f_{2,1}(b,c) - 2f_{2,1}(c,b) + 2(1-\omega)\sum_{k=0}^{K}\sum_{\ell=0}^{L} A_{k\ell}W_{\ell k}^-(c,b) + \frac{\omega}{2\pi}\sum_{i=0}^{I}\sum_{j=0}^{J} C_{ij}W_{ji}^-(c,b) \tag{15d}$$

where functions $W_{ij}^{\pm}(u,v)$ are defined in Appendix D, and functions g_{ij}

$$g_{ij} = \int_0^{\pi/2} \sin^i\theta\cos^j\theta d\theta \tag{16}$$

are available in any standard mathematical handbook.[16]

The divergence of the radiative heat flux is related to the function $G(y,z)$ (see Ref. 8)

$$\bar{\nabla}\cdot\bar{q}^r = (1-\omega)[4\sigma n^2 T^4(y,z) - G(y,z)] \tag{17}$$

where σ is the Stefan-Boltzmann constant, n the refractive index of the medium, and $G(y,z)$ is given by Eq. (12).

Results and Discussion

To illustrate the application of the previous analysis, we consider a rectangular medium, as shown in Fig. 1, subjected to isotropic incident radiation of unit intensity at the boundary surface $z = -c$ and having no energy source within the medium (i.e., $A_{k\ell} = 0$ for $k = 0,\ldots,K$ and $\ell = 0,\ldots,L$).

To solve for the expansion coefficients, we can use either Eqs. (9), obtained by the application of the collocation method, or Eqs. (10), obtained by the application of the Galerkin method. We prefer the Galerkin method because, in general, the solution is more accurate than the collocation method utilizing a uniform spacing of the collocation points. For the representation of the function $G(y,z)$, in a power series as given by Eq. (7a), we consider only the even powers of y because the problem has symmetry about the Oz-axis.

We examine the effect of the order of expansion on the numerical results of $G(y,z)$. Tables 1a and 1b show the results of $G(y,z)$ evaluated along the centerline $y = 0$ for different

Table 1 Results of the incident radiation along the centerline for a unit source at $z = -c$ and no sources within the medium, $b = 0.5$ $[G(0,z)/4\pi, \rho = z/c]$

c	I	J	$\rho=-1$	$\rho=-0.8$	$\rho=-0.6$	$\rho=-0.4$	$\rho=0$	$\rho=0.4$	$\rho=1$
					a) $\omega = 0.1$				
2.5	0	0	0.5012	0.1196	0.0409	0.0174	0.0053	0.0030	0.0013
	2	2	0.5061	0.1262	0.0446	0.0183	0.0025	−0.0006	0.0011
	4	4	0.5083	0.1258	0.0419	0.0159	0.0036	0.0011	0.0004
	6	6	0.5088	0.1250	0.0418	0.0167	0.0033	0.0009	0.0002
0.5	0	0	0.5046	0.3520	0.2654	0.2040	0.1254	0.0809	0.0433
	2	2	0.5088	0.3573	0.2696	0.2065	0.1247	0.0782	0.0412
	4	4	0.5088	0.3571	0.2692	0.2062	0.1248	0.0784	0.0411
	6	6	0.5089	0.3571	0.2692	0.2062	0.1248	0.0784	0.0410
0.05	0	0	0.5048	0.4787	0.4591	0.4420	0.4120	0.3858	0.3504
	2	2	0.5053	0.4792	0.4597	0.4425	0.4125	0.3861	0.3506
	4	4	0.5052	0.4791	0.4595	0.4424	0.4123	0.3859	0.3505
	6	6	0.5052	0.4791	0.4596	0.4424	0.4123	0.3859	0.3505
					b) $\omega = 1$				
2.5	0	0	0.5254	0.1629	0.0875	0.0651	0.0535	0.0507	0.0255
	2	2	0.6099	0.2759	0.1537	0.0838	0.0056	−0.0170	0.0116
	4	4	0.6302	0.2670	0.1182	0.0531	0.0186	0.0076	0.0021
	6	6	0.6328	0.2604	0.1177	0.0595	0.0162	0.0051	0.0008
0.5	0	0	0.5777	0.4527	0.3783	0.3238	0.2500	0.2007	0.1163
	2	2	0.6302	0.5212	0.4357	0.3630	0.2500	0.1710	0.0869
	4	4	0.6291	0.5187	0.4328	0.3605	0.2500	0.1728	0.0862
	6	6	0.6296	0.5193	0.4329	0.3604	0.2500	0.1728	0.0861
			0.522[a]	0.453[a]	0.392[a]	0.342[a]	0.250[a]	0.187[a]	0.117[a]
			0.594[b]	0.510[b]	0.432[b]	0.367[b]	0.251[b]	0.175[b]	0.082[b]
			0.6293[c]	0.5192[c]	0.4329[c]	0.3609[c]	0.2500[c]	0.1729[c]	0.0863[c]
0.05	0	0	0.5550	0.5353	0.5192	0.5042	0.4758	0.4480	0.4006
	2	2	0.5607	0.5421	0.5261	0.5109	0.4814	0.4520	0.4028
	4	4	0.5594	0.5405	0.5243	0.5089	0.4793	0.4500	0.4015
	6	6	0.5600	0.5412	0.5250	0.5097	0.4801	0.4506	0.4018

[a]Results of P_1 approximation in Table 2 of Ref. 2; for $\rho = \pm 0.4$, linear interpolation was used. [b]Results of P_3 approximation in Table 2 of Ref. 2; for $\rho = \pm 0.4$, linear interpolation was used. [c]Results of a numerical solution in Ref. 5, p. 361; except for $\rho = \pm 1$, linear interpolation was used.

Table 2 Results of the incident radiation for a unit source at boundary $z=-c$ and no sources within the medium, $b=c=0.5$ [$G(y,z)/4\pi$, $\rho=z/c$, $\eta=y/b$]

ρ	$\eta=0$	$\eta=0.2$	$\eta=0.4$	$\eta=0.6$	$\eta=0.8$	$\eta=1$
			a) $\omega=0.1$			
1.0	0.0410	0.0406	0.0393	0.0373	0.0346	0.0314
0.4	0.0784	0.0773	0.0742	0.0691	0.0623	0.0543
0.0	0.1248	0.1229	0.1171	0.1075	0.0946	0.0790
−0.4	0.2062	0.2034	0.1945	0.1778	0.1518	0.1176
−0.6	0.2692	0.2665	0.2574	0.2382	0.2020	0.1456
−0.8	0.3571	0.3553	0.3490	0.3336	0.2930	0.1844
−1.0	0.5089	0.5088	0.5084	0.5079	0.5068	−
			b) $\omega=0.5$			
1.0	0.0542	0.0536	0.0518	0.0489	0.0448	0.0396
0.4	0.1062	0.1047	0.1001	0.0928	0.0828	0.0694
0.0	0.1631	0.1605	0.1527	0.1398	0.1219	0.0988
−0.4	0.2559	0.2523	0.2409	0.2197	0.1868	0.1417
−0.6	0.3236	0.3201	0.3084	0.2844	0.2406	0.1711
−0.8	0.4128	0.4103	0.4016	0.3819	0.3338	0.2104
−1.0	0.5512	0.5506	0.5486	0.5452	0.5391	−
			c) $\omega=1.0$			
1.0	0.086	0.085	0.082	0.077	0.069	0.059
0.4	0.173	0.170	0.162	0.149	0.131	0.104
0.0	0.250	0.246	0.233	0.212	0.183	0.142
−0.4	0.360	0.355	0.338	0.307	0.259	0.192
−0.6	0.433	0.428	0.410	0.376	0.317	0.222
−0.8	0.519	0.515	0.501	0.473	0.410	0.260
−1.0	0.630	0.628	0.622	0.613	0.598	−

values of the vertical optical dimension $c=2.5$, 0.5, 0.05 and the single scattering albedo $\omega=0.1$ and 1.0, respectively, with $b=0.5$. Due to the small change in the results between the fourth-order ($I=J=4$) and sixth-order ($I=J=6$) expansion of $G(y,z)$, quite accurate results are obtainable. For comparison purposes, we include in Table 1b the results of a P_1 and P_3 approximation[2] and a numerical solution;[5] the agreement of the results is excellent. For example, for the optical dimensions b and c less than 0.5, a fourth-order expansion in both y and z directions [i.e., 15 terms in the expansion of $G(y,z)$] yields results that are adequate for most engineering applications. However, the larger the optical dimensions of the enclosure (or, in general, the smaller the single scattering albedo), the more terms are needed in the expansion of $G(y,z)$ to obtain the same degree of accuracy.

Tables 2a, 2b, and 2c give our results for $G(y,z)$ at various locations, in a square enclosure, having optical dimensions $b=c=0.5$ for three different values of single scattering albedo $\omega=0.1$, 0.5, and 1, respectively, as calculated from Eq. (12). We note that values of $G(y,z)$ at $z=-c$, $y=\pm b$ are undefined. A comparison of our results in Table 2c for the case of an absorbing, emitting medium with those given by Yuen and Wong[17] shows good agreement in the region near the centerline but a poor agreement near the corners. The reason for this difference is not clear. However, a comparison of Table 2c with Crosbie and Schrenker[5] shows excellent agreement. We believe that the results shown in Table 2 are accurate to within ±1 of the last figure given.

In Fig. 3 we present the results of the reflectivity $[Q_z^-(-c)/(2b\pi)]$, and in Figs. 4 and 5 the respective total heat flow rates $Q_y^+(b)$ and $Q_z^-(c)$ are presented for different optical dimensions of the medium with $\omega=0.1$, 0.5, and 1. An inspection of Fig. 3 reveals that for aspect ratios greater than 5, the reflectivity remains almost unchanged, i.e., it approaches but never reaches the values of the corresponding one-dimensional case. Similarly, as shown in Fig. 4, the total heat flow rate through the side wall $Q_y^+(b)$ is almost unaffected for the aspect ratios greater than about 5. Figure 5 shows that the total heat flow rate through the top wall $Q_z^+(c)$ changes by several orders of magnitude, by the variation of the aspect ratio. However, it appears that $Q_z^+(c)$ increases linearly on the logarithmic scale for larger values of the aspect ratio; this is expected, because the reflectivity and the total heat flow rate $Q_y^+(b)$ remain almost unaffected for larger values of the aspect ratios.

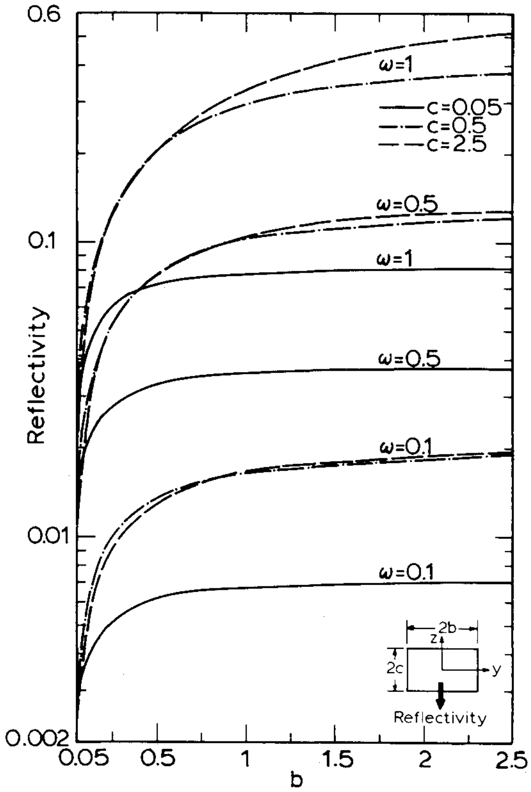

Fig. 3 Reflectivity of the medium for a unit source at $z=-c$.

Our calculations with the Galerkin method achieved an energy balance to within 10^{-8}% for all orders of expansion of $G(y,z)$ for the case $\omega=1$ (no absorption by the medium); with other methods for the case $\omega=1$ (see Refs. 2 and 17), achievable energy balance was within 0.1%.

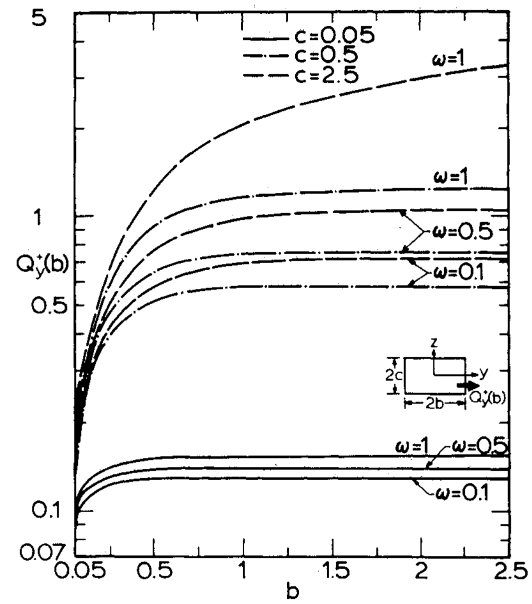

Fig. 4 Total heat flow rate at the boundary $y = b$ for a unit source at $z = -c$ and no sources within the medium.

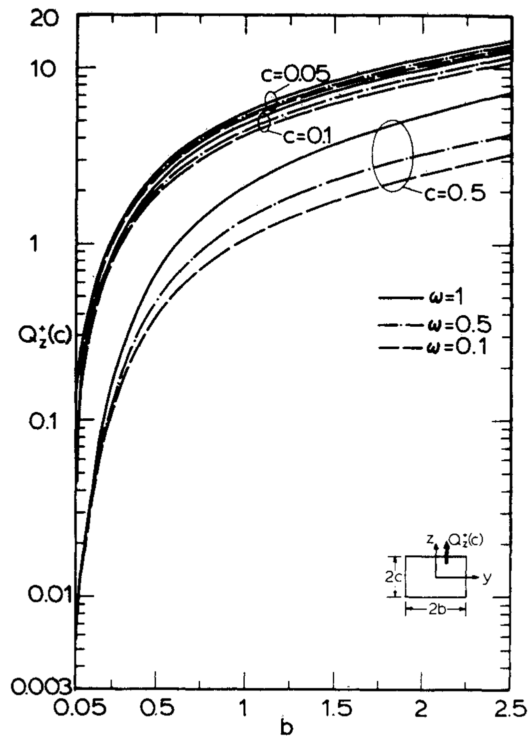

Fig. 5 Total heat flow rate at the boundary $z = c$ for a unit source at $z = -c$ and no sources within the medium.

Conclusion

An efficient method of analysis was presented for solving the radiation heat transfer problem in an absorbing, emitting, isotropically scattering, gray, two-dimensional rectangular medium. By using the expressions developed in this work, the incident radiation, partial heat fluexes in both y and z directions, and total heat flow rates at the boundary surfaces were determined accurately for small, up to moderately large, optical dimensions without much computer time. For example, by using a fourth-order expansion of $G(y,z)$ in both y- and z directions, it took about 120 s of CPU time on the IBM 3081 using the V-level compiler, in double precision, to generate all the values of the incident radiation and the total heat flow rates presented in this work, for a given width and height of the medium. This short execution time makes the present method of analysis viable for the solution of interaction problems, such as combined conduction and radition or combined convection and radiation. The method of solution also has the potential for generalization to problems involving reflection at the boundaries and anisotropic scattering, but to analyze such problems one must first develop the corresponding integral form of the equation of transfer.

Acknowledgment

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Appendix A: Expression for $T_{ij}(y,z)$

The function $T_{ij}(y,z)$ is defined

$$T_{ij}(y,z) = \int_{v=-c}^{c} \int_{u=-b}^{b} K_1(u,y;v,z)u^i v^j \mathrm{d}u\mathrm{d}v \quad \text{(A1)}$$

where $K_1(u,y;v,z)$ is defined by Eq. (5b). $T_{ij}(y,z)$ has the properties

$$T_{ij}(\pm y, \pm z) = (\pm 1)^i (\pm 1)^j T_{ij}(y,z) \quad \text{(A2)}$$

If we define

$$t_{ij}(y,z) = \int_{v=z}^{c} \int_{u=y}^{b} K_1(u,y;v,z)u^i v^j \mathrm{d}u\mathrm{d}v \quad \text{(A3)}$$

then

$$T_{ij}(y,z) = t_{ij}(y,z) + (-1)^i t_{ij}(-y,z) + (-1)^{i+j} t_{ij}(-y,-z) + (-1)^j t_{ij}(y,-z) \quad \text{(A4)}$$

By changing the variables of integration from u,v to polar coordinates, the singularity in Eq. (A3) is eliminated. The integration of Eq. (A3) yields

$$t_{ij}(y,z) = \sum_{k=0}^{i} \sum_{\ell=0}^{j} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix} y^{i-k} z^{j-\ell} \Big\{ (k+\ell)! g_{0,k+\ell+1} g_{k,\ell} - \sum_{m=0}^{k+\ell} \frac{(k+\ell)!}{(k+\ell-m)!} \Big((b-y)^{k+\ell-m} f_{m,\ell}\Big[\frac{1}{2}(c-z), \frac{1}{2}(b-y)\Big] + (c-z)^{k+\ell-m} f_{m,k}\Big[\frac{1}{2}(b-y), \frac{1}{2}(c-z)\Big]\Big)\Big\} \quad \text{(A5)}$$

where g_{ij} is defined by Eq. (16) and $f_{ij}(u,v)$ by Eq. (5a). By using the recurrence relation for $Ki_n(x)$ given by Bickley and Naylor[10]

$$nKi_{n+1}(x) = (n-1)Ki_{n-1}(x) + x[Ki_{n-2}(x) - Ki_n(x)] \quad \text{(A6)}$$

we have developed the following recurrence relations for $f_{i,j}(u,v)$:

$$f_{i+1,j}(u,v) = \frac{1}{j-i-2}\Big[Ki_{i+3}(d)\Big(\frac{2v}{d}\Big)^{i+1}\Big(\frac{u}{v}\Big)^{j-1} + 2vf_{i,j}(u,v) - (j-1)f_{i+1,j-2}(u,v)\Big], \quad j \neq i+2, \quad j > 1 \quad \text{(A7a)}$$

$$f_{i+1,1}(u,v) = \frac{1}{i+1}\Big[Ki_{i+3}(2v) - Ki_{i+3}(d)\Big(\frac{2v}{d}\Big)^{i+1} - 2vf_{i,1}(u,v)\Big], \quad i \geq 0 \quad \text{(A7b)}$$

$$f_{i,i-1}(u,v) = \frac{1}{i}\left[Ki_{i+2}(d)\left(\frac{2u}{d}\right)^i + 2vf_{i-1,i+1}(u,v)\right], \quad i>0 \quad \text{(A7c)}$$

$$f_{i,i+1}(u,v) = \frac{1}{1+i}\left[if_{i-2,i-1}(u,v) + 2vf_{i-3,i-1}(u,v) - (1+2i)f_{i,i-1}(u,v) - 2vf_{i-1,i-1}(u,v) + Ki_{i+2}(d)\left(\frac{2u}{d}\right)^i\right], \quad i>2 \quad \text{(A7d)}$$

where

$$d = 2(u^2+v^2)^{1/2} \quad \text{(A8)}$$

The required starting values of Eqs. (A7), which were approximated using a 40-point Gauss-Legendre quadrature, are $f_{i,0}(u,v)$, $i=0,\ldots,i_{\max}$, $f_{0,1}(u,v)$, $f_{1,2}(u,v)$, and $f_{2,3}(u,v)$. However, it is possible to use the recently derived infinite series expression for $Ki_1(x)$ (see Ref. 18) in the integration of Eq. (A3), which converges rapidly for $x \le 1$. By using this infinite series, we obtain

$$t_{ij}(y,z) = \sum_{k=0}^{i}\sum_{\ell=0}^{j}\binom{i}{k}\binom{j}{\ell} y^{i-k}z^{j-\ell}\left\{\frac{\pi/2}{k+\ell+1} \times [(b-y)^{k+\ell+1}\Phi_{1,\ell}(c-z,b-y) + (c-z)^{k+\ell+1}\Phi_{1,k}(b-y,c-z)] - (b-y)^{k+\ell+1}P_{k,\ell}(c-z,b-y) - (c-z)^{k+\ell+1}P_{\ell,k}(b-y,c-z)\right\} \quad \text{(A9)}$$

where

$$P_{i,j}(u,v) = \sum_{m=0}^{\infty}\frac{v(v/2)^{2m}}{(2m+1)(m!)^2(2m+i+j+2)} \times \left[\left(\psi_{m+1} + \frac{1}{2m+1} + \frac{1}{2m+i+j+2}\right)\Phi_{2m+2,j}(u,v) - \Upsilon_{2m+2,j}(u,v)\right] \quad \text{(A10a)}$$

$$\psi_n = -\gamma, \quad n=1 \quad \text{(A10b)}$$

$$\psi_n = -\gamma + \sum_{m=1}^{n-1}\frac{1}{m}, \quad n>1 \quad \text{(A10c)}$$

$$\Phi_{k,\ell}(u,v) = \int_{\theta=0}^{\tan^{-1}(u/v)} \sec^k\theta\tan^\ell\theta d\theta, \quad k,\ell \ge 0 \quad \text{(A10d)}$$

and

$$\Upsilon_{k,\ell}(u,v) = \int_{\theta=0}^{\tan^{-1}(u/v)} \sec^k\theta\tan^\ell\theta \ln(\sec\theta) d\theta, \quad k,\ell \ge 0 \quad \text{(A10e)}$$

where $\gamma = 0.577216\ldots$ is Euler's constant.

Recurrence relations for $\Phi_{k,\ell}(u,v)$ are available in standard mathematical handbooks. However, we derived the following recurrence relations for $\Upsilon_{k,\ell}(u,v)$:

$$\Upsilon_{k,\ell}(u,v) = \frac{1}{\ell+k+1}\left[\left(\frac{d}{2v}\right)^{k-2}\ln\left(\frac{d}{2v}\right)\left(\frac{u}{v}\right)^{\ell+1} + (k-2)\Upsilon_{k-2,\ell}(u,v) - \Phi_{k-2,\ell+2}(u,v)\right] \quad \text{(A11a)}$$

and

$$\Upsilon_{2,\ell}(u,v) = \Upsilon_{4,\ell-2}(u,v) - \Upsilon_{2,\ell-2}(u,v) \quad \text{(A11b)}$$

If $b \le 2$, and $c \le 2$, the evaluation of $t_{ij}(y,z)$ is much more rapid by using Eq. (A9) than (A5).

Appendix B: Expressions for $T_{ij}^{\mu\nu}$ and $H^{\mu\nu}$

We define

$$T_{ij}^{\mu\nu} = \int_{z=-c}^{c}\int_{y=-b}^{b} T_{ij}(y,z)y^\mu z^\nu dydz \quad \text{(B1)}$$

By performing the integration, we obtain

$$T_{ij}^{\mu\nu} = [1+(-1)^{i+\mu}+(-1)^{j+\nu}+(-1)^{i+j+\mu+\nu}] \times \sum_{k=0}^{i}\sum_{\ell=0}^{j}\sum_{m=0}^{\mu+i-k}\sum_{n=0}^{\nu+j-\ell}\binom{i}{k}\binom{j}{\ell}\binom{\mu+i-k}{m}\binom{\nu+j-\ell}{n}$$
$$(-1)^{m+n}b^{\mu+i-k-m}c^{\nu+j-\ell-n}\left\{(k+\ell)!g_{0,k+\ell+1}\frac{(2b)^{m+1}(2c)^{n+1}}{(n+1)(m+1)} \times g_{k,\ell} - \sum_{s=0}^{k+\ell}\frac{(k+\ell)!}{(k+\ell-s)!}\right.$$
$$\times\left[\frac{(2c)^{n+1}}{n+1}(k+\ell+m-s)!g_{0,k+\ell+m+2}g_{\ell,k+m+1} + \frac{(2b)^{m+1}}{m+1}(k+\ell+n-s)!g_{0,k+\ell+n+2}g_{k,\ell+n+1} - \frac{(m+n+2)}{(m+1)(n+1)}(\alpha-s)!g_{0,\alpha+2}\right.$$
$$\times g_{\ell+n+1,k+m+1} - \frac{(2c)^{n+1}}{n+1}\sum_{t=0}^{k+\ell+m-s}\frac{(k+\ell+m-s)!}{(k+\ell+m-s-t)!} \times (2b)^{k+\ell+m-s-t}f_{s+t+1,\ell}(c,b) + (2c)^{k+\ell+m-s-t}f_{s+t+1,k+m+1}(b,c)$$
$$-\frac{(2b)^{m+1}}{m+1}\sum_{t=0}^{k+\ell+n-s}\frac{(k+\ell+n-s)!}{(k+\ell+n-s-t)!} \times (2c)^{k+\ell+n-s-t}f_{s+t+1,k}(b,c) + (2b)^{k+\ell+n-s-t}f_{s+t+1,\ell+n+1}(c,b)$$
$$\left.\left. + \frac{(m+n+2)}{(m+1)(n+1)}\sum_{t=0}^{\alpha-s}\frac{(\alpha-s)!}{(\alpha-s-t)!} + (2b)^{\alpha-s-t}f_{s+t+1,\ell+n+1} \times (c,b) + (2c)^{\alpha-s-t}f_{s+t+1,k+m+1}(b,c)\right]\right\} \quad \text{(B2)}$$

where

$$\alpha = k+\ell+m+n+1 \quad \text{(B3)}$$

Next, we define

$$H^{\mu\nu} = 2[1+(-1)^\mu]\int_{z=-c}^{c}\int_{y=-b}^{b} f_{0,0}\left[\frac{1}{2}(b+y), \frac{1}{2}(c+z)\right]y^\mu z^\nu dydz \quad \text{(B4)}$$

The integration is carried out to obtain

$$H^{\mu\nu}=2[1+(-1)^{\mu}]\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}\binom{\mu}{i}\binom{\nu}{j}$$

$$\times b^{\mu-i}c^{\nu-j}\frac{(-1)^{\mu+\nu+i+j}}{i+1}\Big\{(2b)^{i+1}j!g_{0,j+2}g_{0,j+1}$$

$$-(i+j+1)!g_{0,i+j+3}g_{i+1,j+1}-(2b)^{i+1}\sum_{k=0}^{j}\frac{j!}{(j-k)!}$$

$$\times[(2c)^{j-k}f_{k+1,0}(b,c)+(2b)^{j-k}f_{k+1,j+1}(c,b)]$$

$$+\sum_{k=02}^{i+j+1}\frac{(i+j+1)!}{(i+j+1-k)!}[(2c)^{i+j+1-k}f_{k+1,i+1}(b,c)$$

$$+(2b)^{i+j+1-k}f_{k+1,j+1}(c,b)]\Big\} \tag{B5}$$

Appendix C: Expressions for $Z_{ij}^{\mp}(y,z)$ and $Y_{ij}^{\mp}(y,z)$

We define

$$Z_{ij}^{\mp}(y,z)=(\mp 1)^{i+j}U_{ij}(\mp y,\mp z)$$

$$+(\pm 1)^{i}(\mp 1)^{j}U_{ij}(\pm y,\mp z) \tag{C1a}$$

and

$$Y_{ij}^{\mp}(y,z)=(\mp 1)^{i+j}U_{ij}^{*}(\mp y,\mp z)$$

$$+(\mp 1)^{i}(\pm 1)^{j}U_{ij}^{*}(\mp y,\pm z) \tag{C1b}$$

the $nU_{ij}(y,z)$ and $U_{ij}^{*}(y,z)$ are determined from

$$U_{ij}(y,z)=\sum_{k=0}^{i}\sum_{\ell=0}^{j}\binom{i}{k}\binom{j}{\ell}y^{i-k}z^{j-\ell}$$

$$\times[(k+\ell)!g_{0,k+\ell+2}g_{\ell+1,k}-V_{k\ell}(b-y,c-z)] \tag{C2a}$$

and

$$U_{ij}^{*}(y,z)=\sum_{k=0}^{i}\sum_{\ell=0}^{j}\binom{i}{k}\binom{j}{\ell}y^{i-k}z^{j-\ell}$$

$$\times[(k+\ell)!g_{0,k+\ell+2}g_{\ell,k+1}-V_{\ell k}(c-z,b-y)] \tag{C2b}$$

where

$$V_{ij}(u,v)=\sum_{k=0}^{i+j}\frac{(i+j)!}{(i+j-k)!}[u^{i+j-k}f_{k+1,j+1}(v/2,u/2)$$

$$+v^{i+j-k}f_{k+1,i}(u/2,v/2)] \tag{C3}$$

Appendix D: Expression for $W_{ij}^{\mp}(u,v)$

The explicit expression for $W_{ij}^{\mp}(u,v)$ is given by

$$W_{ij}^{\mp}(u,v)=[(\mp 1)^{i+j}$$

$$+(\pm 1)^{i}(\mp 1)^{j}]\sum_{k=0}^{i}\sum_{\ell=0}^{j}\binom{i}{k}\binom{j}{\ell}(-v)^{j-\ell}(k+\ell\}!$$

$$\times\Big[g_{0,k+\ell+2}g_{\ell+1,k}u^{i-k+1}\frac{1+(-1)^{i+k}}{i-k+1}$$

$$-\sum_{s=0}^{k+\ell}\sum_{m=0}^{i-k}\binom{i-k}{m}\frac{(-1)^{m}u^{i-k-m}}{(k+\ell-s)!}$$

$$\times\Big\{(m+k+\ell-s)!g_{0,m+k+\ell+3}g_{\ell+1,m+k+1}$$

$$+\frac{(2v)^{k+\ell-s}}{m+1}\Big((2u)^{m+1}f_{s+1,k}(u,v)$$

$$-(2v)^{m+1}f_{s+1,m+k+1}(u,v)\Big)$$

$$-\sum_{t=0}^{m+k+\ell-s}\frac{(m+k+\ell-s)!}{(m+k+\ell-s-t)!}(2u)^{m+k+\ell-s-t)!}f_{s+t+2,\ell+1}$$

$$\times(v,u)+(2v)^{m+k+\ell-s-t}f_{s+t+2,m+k+1}(u,v)\Big\}\Big] \tag{D1}$$

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